

Note

Alternative to Padé Technique for Iterative Solution of Integral Equations

It is well known that when the kernel of Fredholm integral equations of the second kind has the absolute magnitude of one or more eigenvalues greater than unity the Neumann series of such equations diverge. Padé technique is a practical way for constructing the correct solution from the divergent Neumann series. Here we propose an alternative method for the iterative solution of such equations, which relies on writing auxiliary integral equations with "weaker" kernels so that the auxiliary equations have a convergent Neumann series. Then the solution of the original equation is expressed in terms of the (convergent Neumann series) solution of the auxiliary equations. The kernel of the auxiliary equations can systematically be made weaker and the rate of convergence of the Neumann series of the auxiliary equations improved in order to make the method more useful. The method is illustrated numerically in certain cases.

1. INTRODUCTION

In this paper we develop a practical method for the iterative solution of Fredholm integral equations [1] of the second kind for the unknown function $y(x)$,

$$y(x) = f(x) + \int_a^b K(x, \xi) y(\xi) d\xi, \quad a \leq x \leq b, \quad (1)$$

or in schematic operator notation

$$y = f + Ky, \quad (2)$$

where y and f are real or complex valued functions in $L^2(a, b)$ and K is a square integrable kernel. As the equation is of the Fredholm type, the Fredholm alternative is valid and Eq. (1) can be uniformly approximated by a matrix equation of finite rank, which can be solved by matrix inversion. But in the case of a realistic and complicated problem the dimension of the resultant matrix equation could be large and inversion of a large matrix is not a trivial numerical task. A simple alternative (to the method of matrix inversion) is the Liouville-Neumann method of successive substitutions [1], which is commonly known as the iterative method for the solution of Eq. (1). Such iterative method works for sufficiently weak kernel K . In mathematical language the so-called Neumann series converges if and only if all the eigenvalues η_ν of the kernel K , defined by [1, 2]

$$K\psi_\nu = \eta_\nu \psi_\nu, \quad (3)$$

have absolute values smaller than unity,

$$|\eta_\nu| < 1. \quad (4)$$

In most of the physically interesting problems the above condition is not satisfied and the Neumann series of Eq. (1) diverges. But one can use the technique of Padé approximation [3] to solve integral equation (1) by extracting information from its Neumann series, even if the series diverges. Although the Padé technique has become a useful and efficient tool for solving Eq. (1), it has its limitations also. The number of terms of the Neumann series needed to construct accurate Padé approximants could be large enough to render the method unattractive in practice. Secondly, there are questions about the uniformity of convergence of Padé approximants and in particular about the occurrence of poles in the Padé approximants, which are not related to the solution of Eq. (1).

Here we propose a method for iterative solution of Eq. (1). The method relies on writing auxiliary equation(s) whose kernel has eigenvalues satisfying condition (4). Then the auxiliary equations have convergent Neumann series solutions. The solution of the original Eq. (1) is then expressed in terms of the solutions of the auxiliary equations. The convergence properties of the Neumann series for the auxiliary equations of the present method can be improved systematically. The present method is simple to use and can be an efficient method for solving Eq. (1).

The method is based on the following simple algebraic manipulations. The equation we would like to solve is written as

$$y_1^{(0)}(x) = K_0(x) + \int K_1(x, \xi) y_1^{(0)}(\xi) d\xi, \quad a \leq x \leq b, \quad (5)$$

where in Eq. (5) and in the rest of the paper the limits of integrations are from a to b unless otherwise specified. The notation in Eq. (5) is slightly changed from that of Eq. (1) for the sake of future convenience. In Eq. (5) $y_1^{(0)}$ is the unknown function, K_0 is the inhomogeneous term and K_1 is the kernel. Next we introduce the following auxiliary equations

$$y_2^{(0)}(x) = K_0(x) + \int K_2(x, \xi) y_2^{(0)}(\xi) d\xi$$

and (6)

$$y_2^{(1)}(x) = K_1(x, \xi_1) + \int K_2(x, \xi) y_2^{(1)}(\xi) d\xi$$

with

$$K_2(x, \xi) = K_1(x, \xi) - K_1(x, \xi_1) \gamma_1(\xi), \quad (7)$$

where ξ_1 is a point conveniently chosen in (a, b) and $\gamma_1(\xi)$ is an arbitrary flexible function. Using Eq. (7), Eq. (5) can be written as

$$y_1^{(0)}(x) = K_0(x) + K_1(x, \xi_1) \mathcal{I}_1 + \int K_2(x, \xi) y_1^{(0)}(\xi) d\xi \quad (8)$$

with

$$\mathcal{I}_1 = \int d\xi \gamma_1(\xi) y_1^{(0)}(\xi). \quad (9)$$

From Eqs. (6) and (8) we have

$$y_1^{(0)}(x) = y_2^{(0)}(x) + y_2^{(1)}(x) \mathcal{I}_1, \quad (10)$$

which is the key equation of the present method. Equation (10) is still an integral equation in $y_1^{(0)}$ but with a separable kernel and hence can be solved "analytically." The solution of Eq. (10) is given by

$$y_1^{(0)}(x) = y_2^{(0)}(x) + y_2^{(1)}(x) \frac{\int d\xi \gamma_1(\xi) y_2^{(0)}(\xi)}{1 - \int d\xi \gamma_1(\xi) y_2^{(1)}(\xi)}. \quad (11)$$

Equation (11) uses the solution of two auxiliary integral equations—Eqs. (6)—with kernel K_2 given by Eq. (7). If the function $\gamma_1(\xi)$ of Eq. (7) is conveniently chosen so that the second term on the right-hand side of Eq. (7) is a good approximation to $K_1(x, \xi)$, then $K_2(x, \xi)$ will be much weaker than $K_1(x, \xi)$ of Eq. (5). Under these circumstances we can hope that Eqs. (6) will have convergent Neumann iterative series solutions.

The present method is based on the convergent Neumann series solution for Eqs. (6) and the construction of the solution of Eq. (5) through Eq. (11). If with the best choice for $\gamma_1(\xi)$ the Neumann series for Eqs. (6) diverges or converges slowly we shall show how to improve the rate of convergence of the method systematically.

In Section 2 we critically discuss the method presented in this section and show how one can systematically improve the method. In Section 3 we illustrate the method numerically in certain cases and finally in Section 4 we give a brief summary and concluding remarks.

2. THE METHODS

A. Definition of the Basic Method

In this subsection we critically discuss the method presented in Section 1. Although the basic equations of the method have been described in Section 1 we still have to define the method properly, because so far we have said nothing about how to choose the function $\gamma_1(\xi)$ of Eq. (7). As has been explained in Section 1 the success of the

method depends on a correct choice of γ_1 . For a sufficiently small operator norm $\|K_2\|$, Eqs. (6) will have convergent Neumann series. Though we are not directly concerned about eigenvalues of K_2 , it should be remembered that convergent Neumann series for Eqs. (6) imply that all the eigenvalues of K_2 have their absolute magnitude smaller than unity and vice versa [1, 2]. So we should choose γ_1 so that K_2 satisfies this criteria. There is another restriction on the choice of γ_1 . The function γ_1 should be so chosen that all the integrals in the method remain well defined and finite. We shall not present a rigorous theory about how to choose γ_1 so that the rate of convergence of the Neuman series of Eqs. (6) is the fastest. But we choose γ_1 based on simple intuitive arguments in order to have satisfactory rate of convergence.

An observation which helps in the choice of γ_1 is that if

$$\gamma_1(\xi_1) = 1, \tag{12}$$

K_2 defined by (7) satisfies $K_2(x, \xi_1) = 0$. K_2 will be small at other points if γ_1 is conveniently chosen. In this paper we shall mainly be concerned with two different choices of $\gamma_1(\xi)$.

First we choose:

$$\text{Choice A: } \gamma_1(\xi) = \frac{K_1(\xi_1, \xi)}{K_1(\xi_1, \xi_1)}. \tag{13}$$

This choice of $\gamma_1(\xi)$ satisfies Eq. (12) and $K_2(x, \xi)$ defined by Eq. (7) is zero for $x = \xi_1$ or $\xi = \xi_1$. If $x = \xi_1$ is the point where $K_1(x, x)$ is large $K_2(x, \xi)$ will be zero at $x = \xi_1$ or $\xi = \xi_1$. This choice of γ_1 will make $K_2(x, \xi)$ small everywhere because the second term on the right-hand side of Eq. (7) is the Bateman approximation [4] for $K_1(x, \xi)$, which is exact around the point $x = \xi_1$ or $\xi = \xi_1$ and is expected to be a good separable representation to K_1 over the whole domain of x and ξ . Of course, we are not claiming that choice (13) for γ_1 is the ideal choice.

A better choice of $\gamma_1(\xi)$ could be found if we recall the fact that we would like to make the magnitude of $K_2(x, \xi)$ as small as possible for all ξ . The choice of γ_1 given by Eq. (13) is designed to make the second term on the right hand side of Eq. (7) a good approximation to the first term, e.g., $K_1(x, \xi)$, in a pointwise sense. By doing this we are demanding too much from $\gamma_1(\xi)$ in view of the fact that K_1 is in general a nondegenerate kernel and a simple Bateman approximation is not the best approximation to it. A better and weaker condition can be imposed on $\gamma_1(\xi)$, which requires the magnitude $|K_2(x, \xi)|$ to be as small as possible for all x . It is clear from Eq. (6) that such a choice will improve the rate of convergence of the Neumann series. A simple way to achieve this is to demand that $[K_2(x, \xi)]^2$ has a minimum as a function of $\gamma_1(\xi)$. As $\gamma_1(\xi)$ is independent of the parameter x we prefer to integrate $[K_2(x, \xi)]^2$ in variable x with respect to some weight function $\omega_1(x)$ and demand

$$\delta \int dx \omega_1(x) [K_2(x, \xi)]^2 = 0, \tag{14}$$

where δ denotes variations with respect to $\gamma_1(\xi)$. Choice *A* of γ_1 minimizes the integrand $K_2(x, \xi)$ of Eq. (14) in a pointwise sense and minimizing the integrand in a pointwise sense is not the optimal way to minimize an integral. The optimal way is to satisfy Eqs. (7) and (14), which immediately yields the following choice for γ_1

$$\text{Choice B: } \gamma_1(\xi) = \frac{\int dx \omega_1(x) K_1(x, \xi) K_1(x, \xi_1)}{\int dx \omega_1(x) K_1(x, \xi_1) K_1(x, \xi_1)}. \quad (15)$$

This choice of γ_1 also satisfies Eq. (12) and hence enjoys the associated advantages. It is easy to check that the second derivative of the integral in Eq. (14) with respect to $\gamma_1(\xi)$ is positive if $\omega_1(x)$ is a positive function in (a, b) . So $\gamma_1(\xi)$ given by (15) will really give us the desired minimum. We take a very simple form for $\omega_1(x)$, e.g., $\omega_1(x) = x^n$, in our numerical examples, where n is a small positive or negative integer provided that the integrals in (15) are finite with this choice of $\omega_1(x)$. Such an $\omega_1(x)$ suppresses or enhances parts of the integral in (15) and generates a wide class of $\gamma_1(\xi)$.

Both these choices—choices *A* and *B* of $\gamma_1(\xi)$ —have another advantage that all the integrals in Eqs. (6)–(11) remain finite and well defined with these choices. The only arbitrariness we have now is in the choice of the point ξ_1 in both the cases and in the choice of the function $\omega_1(x)$ in Choice B. We shall see in our numerical studies that this arbitrariness can be turned to good advantage—we can vary ξ_1 and ω_1 in order to obtain the best convergence of the Neumann series of Eq. (6).

B. Improvement of Convergence

If the rate of convergence of Eqs. (6) is not satisfactory we may apply the reduction procedure of Section 1 once again to Eq. (5) in order to have auxiliary equations with better convergence properties. For this purpose we introduce the following auxiliary equations

$$\begin{aligned} y_3^{(0)}(x) &= K_0(x) + \int K_3(x, \xi) y_3^{(0)}(\xi) d\xi, \\ y_3^{(1)}(x) &= K_1(x, \xi_1) + \int K_3(x, \xi) y_3^{(1)}(\xi) d\xi, \end{aligned} \quad (16)$$

and

$$y_3^{(2)}(x) = K_2(x, \xi_2) + \int K_3(x, \xi) y_3^{(2)}(\xi) d\xi$$

with

$$K_3(x, \xi) = K_2(x, \xi) - K_2(x, \xi_2) \gamma_2(\xi), \quad (17)$$

where ξ_2 is a point conveniently chosen in (a, b) and $\gamma_2(\xi)$ is an arbitrary flexible function. Using Eqs. (7) and (17), Eq. (5) can be written as

$$y_1^{(0)}(x) = K_0(x) + K_1(x, \xi_1) \mathcal{J}_1 + K_2(x, \xi_2) \mathcal{J}_2 + \int K_3(x, \xi) y_1^{(0)}(\xi) d\xi, \tag{18}$$

with

$$\mathcal{J}_2 = \int d\xi \gamma_2(\xi) y_1^{(0)}(\xi) \tag{19}$$

and \mathcal{J}_1 defined by Eq. (9). From Eqs. (16) and (18) we have

$$y_1^{(0)}(x) = y_3^{(0)}(x) + y_3^{(1)}(x) \mathcal{J}_1 + y_3^{(2)}(x) \mathcal{J}_2. \tag{20}$$

Using the definitions (9) and (19) it is easy to see that \mathcal{J}_1 and \mathcal{J}_2 satisfy the following set of algebraic equations

$$\begin{aligned} \mathcal{J}_1 &= \int dx \gamma_1(x) [y_3^{(0)}(x) + y_3^{(1)}(x) \mathcal{J}_1 + y_3^{(2)}(x) \mathcal{J}_2], \\ \mathcal{J}_2 &= \int dx \gamma_2(x) [y_3^{(0)}(x) + y_3^{(1)}(x) \mathcal{J}_1 + y_3^{(2)}(x) \mathcal{J}_2]. \end{aligned} \tag{21}$$

Equations (20) and (21) use the solutions of the auxiliary equations (16) with the kernel K_3 defined by Eq. (17). We have seen how to make K_2 weaker than K_1 with a proper choice of γ_1 . Similarly we hope to make K_3 weaker than K_2 with a proper choice of γ_2 . As before, the function γ_2 should be so chosen that all the integrals in the method remain well defined and finite.

As before, we consider the following two choices of γ_2 :

$$\text{Choice A: } \gamma_2(\xi) = \frac{K_2(\xi_2, \xi)}{K_2(\xi_2, \xi_2)}, \tag{22}$$

and

$$\text{Choice B: } \gamma_2(\xi) = \frac{\int dx \omega_2(x) K_2(x, \xi) K_2(x, \xi_2)}{\int dx \omega_2(x) K_2(x, \xi_2) K_2(x, \xi_2)}, \tag{23}$$

which follow immediately from Eqs. (13) and (15) for γ_1 . Here $\omega_2(x)$ is again a weight function. If we choose γ_1 according to (13) and γ_2 according to (22) we immediately have the following desirable property for K_3 : $K_3(x, \xi) = 0$, when $x = \xi_1$ or ξ_2 or $\xi = \xi_1$ or ξ_2 . The point $x = \xi_1$ is chosen where $K_1(x, x)$ is large. Similarly we

choose $x = \xi_2$ where $K_2(x, x)$ is large. For reasons explained after Eq. (13) K_3 will be much smaller than K_2 in the entire domain (a, b) . Hence the kernel $K_3(x, \xi)$ apart from being zero for x or $\xi = \xi_1$ or ξ_2 is also much smaller than K_1 and hence Eqs. (16) are expected to have a convergent Neumann series.

On the other hand if we choose γ_1 according to Eq. (15) and γ_2 according to Eq. (23) we only have $K_3(x, \xi) = 0$, when $\xi = \xi_2$ and *not* when $x = \xi_1$ or ξ_2 or $\xi = \xi_1$. But this choice (called Choice B) is more efficient than the choice A in yielding a rapidly convergent Neumann series as we shall see in the following.

If $K_1(x, \xi)$ is not a smooth function over the domain (a, b) of x and ξ and if the typical value of $K_1(x, \xi)$ is large compared to unity, then Eqs. (16) may not have a convergent Neumann series. But now it is not difficult to make a generalization of the method by introducing successive subtractions in the kernel till we have a convergent Neumann series for the auxiliary equations.

C. Generalization

The generalization of the discussion of the previous subsections is simple to implement in practice. We introduce the final kernel K_N through successive subtractions

$$K_{i+1}(x, \xi) = K_i(x, \xi) - K_i(x, \xi_i) \gamma_i(\xi), \quad i = 1, 2, \dots, N-1. \quad (24)$$

The auxiliary equations are introduced by

$$\begin{aligned} y_N^{(0)}(x) &= K_0(x) + \int K_N(x, \xi) y_N^{(0)}(\xi) d\xi, \\ y_N^{(1)}(x) &= K_1(x, \xi_1) + \int K_N(x, \xi) y_N^{(1)}(\xi) d\xi, \\ &\vdots \\ y_N^{(i)}(x) &= K_i(x, \xi_i) + \int K_N(x, \xi) y_N^{(i)}(\xi) d\xi, \\ &\vdots \\ y_N^{(N-1)}(x) &= K_{N-1}(x, \xi_{N-1}) + \int K_N(x, \xi) y_N^{(N-1)}(\xi) d\xi. \end{aligned} \quad (25)$$

Using Eq. (24), Eq. (5) can be written as

$$y_1^{(0)}(x) = K_0(x) + \sum_{i=1}^{N-1} K_i(x, \xi_i) \mathcal{J}_i + \int K_N(x, \xi) y_1^{(0)}(\xi) d\xi, \quad (26)$$

with

$$\mathcal{J}_i = \int d\xi \gamma_i(\xi) y_1^{(0)}(\xi). \quad (27)$$

From Eqs. (25) and (26) we have

$$y_1^{(0)}(x) = y_N^{(0)}(x) + \sum_{i=1}^{N-1} y_N^{(i)}(x) \mathcal{J}_i, \tag{28}$$

with \mathcal{J}_i satisfying the following set of linear equations:

$$\mathcal{J}_j = \int d\xi \gamma_j(\xi) \left[y_N^{(0)}(\xi) + \sum_{i=1}^{N-1} \mathcal{J}_i y_N^{(i)}(\xi) \right]. \tag{29}$$

Equations (28) and (29) use the solution of the auxiliary equations (25) with the operators K_i defined by Eq. (24). We have seen in the last subsection how to make the kernels K_3 and K_2 successively weaker in comparison with K_1 . So with proper choice of γ_i K_i is expected to be much weaker than K_1 . As before, the functions γ_i should be so chosen that all the integrals in the method remain well defined and finite.

Following the previous subsections we consider the following choices for γ_i ,

$$\text{Choice A: } \gamma_i(\xi) = \frac{K_i(\xi_i, \xi)}{K_i(\xi_i, \xi_i)}. \tag{30}$$

$$\text{Choice B: } \gamma_i(\xi) = \frac{\int dx \omega_i(x) K_i(x, \xi) K_i(x, \xi_i)}{\int dx \omega_i(x) K_i(x, \xi_i) K_i(x, \xi_i)}. \tag{31}$$

If we choose γ_i according to (30) we immediately have the following desirable property for K_N : $K_N(x, \xi) = 0$, when $x = \xi_1$ or ξ_2 or ξ_3 or \dots ξ_{N-1} , or $\xi = \xi_1$ or ξ_2 or ξ_3 or \dots ξ_{N-1} .

3. NUMERICAL CALCULATIONS

We consider the numerical solution of

$$y(x) = K_0(x) + \lambda \int_0^1 K_1(x, \xi) y(\xi) d\xi, \tag{32}$$

where the kernel K_1 will be assumed to have the two following forms:

$$K_1(x, \xi) = 0.1(x + \xi + 1)^5, \tag{33}$$

and

$$K_1(x, \xi) = (x + \xi)^6(x + \xi + 1)^{-1}. \tag{34}$$

The success of the method lies in a convergent Neumann series for Eq. (25). So in this section we study the convergence properties of such equations—in particular that

of Eq. (6) after one subtraction. For this purpose we approximate the integral in Eq. (32) by a discrete sum by using 16-point Gauss quadratures between 0 and 1. We shall study the convergence properties of the following series

$$S_i^{(l)} = \sum_{j=1}^l \lambda^j K_i^j, \quad i = 1, 2, \quad (35)$$

where K_1 is defined by Eqs. (33) and (34) and K_2 by Eq. (7). The convergence of the series (35) will determine the convergence of the method. For example, for $i = 1$ (35) determines the rate of convergence of the original equation (5). For $i = 2$ it determines the rate of convergence of the once subtracted auxiliary equations (6) and so on.

We studied the rate of convergence of Eqs. (5) and (6) for the above kernels for a wide range of values of λ and the variables ξ_1 and $\omega_1(x) = x^n$ in Eqs. (13) and (15) and reached the following general conclusions. As expected convergence of Eq. (6) is more difficult to obtain when λ is large. Choice A of $\gamma_1(\xi)$ given by Eq. (13) was good for obtaining a rapidly convergent Neumann series of Eq. (6) for small λ ($\lambda \lesssim 5$). But for large λ ($\lambda \gtrsim 10$) Choice B of $\lambda_1(\xi)$ given by Eq. (15) was by far superior to choice A in finding a rapidly convergent Neumann series. In the case of

TABLE I

		$S_i'(x, 0.9648)$			
i	x	$I = 2$	$I = 4$	$I = 6$	$I = 8$
1	0.9856	0.375(3)	0.148(7)	0.587(10)	0.233(14)
2		-0.3550	-0.3596	-0.3596	-0.3596
1	0.8494	0.290(3)	0.115(7)	0.455(10)	0.180(14)
2		0.0192	0.0196	0.0196	0.0196
1	0.5877	0.171(3)	0.678(6)	0.268(10)	0.106(14)
2		0.3819	0.3867	0.3867	0.3867
1	0.2393	0.768(2)	0.305(6)	0.121(10)	0.478(13)
2		0.4311	0.4361	0.4361	0.4361

Note. $S_i'(x, 0.9648)$ for various x and I for the kernel given by Eq. (33). $i = 1$ refer to the Neumann series for the original kernel and $i = 2$ refer to the once subtracted kernel. We used $\xi_1 = 0.3319$ and $\omega_1(x) = x^2$. This ξ_1 was a point of the integration mesh. The number in the parenthesis gives the exponent of 10 which multiplies the associated expression.

Choice B the results were not strongly sensitive to $\omega_1(x)$ ($=x^n$), provided that n was a small positive integer. n can not be negative in this case because such an n will lead to divergent integrals in Eq. (15) with K_1 given by (33) and (34). Finally we find that our results are moderately sensitive to ξ_1 .

We show numerical results only for large λ and for the choice B of γ_1 . All values of n in $\omega_1(x) = x^n$ ($0 < n < 4$) gave good convergence but the results for $n = 2$ appeared to be slightly better than others. We show numerical results in Tables I and II for Choice B of γ_1 with $\omega_1(x) = x^2$. Some experimentation was done to choose the best ξ_1 in each case and it was found that the best ξ_1 was always near the middle of the range of integrations (0, 1) rather than near its ends. Table I shows the elements $S_i^j(x, 0.9648)$ for the series (35) for the kernel defined by (33) for various x and I and for $i = 1, 2$. We used $\xi_1 = 0.3319$, $\omega_1(x) = x^2$, and $\lambda = 10$. With this λ the largest eigenvalue μ of the kernel given by (33) is given by $\mu = 62.9$. This means that the Neumann series of the original equation is diverging very rapidly, the ratio of successive terms in the Neumann series after a large number of iteration being equal to μ . Table II shows the same elements of (35) but for the kernel defined by (34) for $\xi_1 = 0.5069$, $\omega_1(x) = x^2$, and $\lambda = 10$. In this case $\mu = 34.9$. From Tables I and II we find that the convergence is satisfactory. The other elements of the series (35) converge equally rapidly.

TABLE II

		$S_i^j(x, 0.9648)$			
i	x	$I = 2$	$I = 4$	$I = 6$	$I = 8$
1	0.9856	0.175(3)	0.214(6)	0.260(9)	0.317(12)
2		-0.3352	-0.3505	-0.3511	-0.3512
1	0.8494	0.116(3)	0.141(6)	0.172(9)	0.209(12)
2		0.1221	0.1278	0.1281	0.1281
1	0.5877	0.468(2)	0.572(5)	0.696(8)	0.848(11)
2		0.3570	0.3722	0.3729	0.3730
1	0.2393	0.105(2)	0.128(5)	0.156(8)	0.191(11)
2		0.1817	0.1905	0.1909	0.1909

Note. Same as in Table I for the kernel given by Eq. (34) and for $\xi_1 = 0.5069$ and $\omega_1(x) = x^2$.

4. DISCUSSION

In this paper we propose a new method for solving Fredholm integral equations of the second kind. The present method is an iterative method and is simple to implement in practice. After solving Eqs. (25) by iteration one still has to solve Eq. (28) with degenerate kernel. So the present method should be considered as a combination of the degenerate kernel method and the Liouville–Neumann method of successive substitutions.

We study the method numerically for both the choices of the arbitrary function γ and find that both of them give satisfactory convergence after one subtraction if the rate of divergence of the original equation is not too large, i.e., when the largest eigenvalue of the kernel $\mu \lesssim 10$. But when the Neumann series of the original equation diverges very strongly the choice B of γ was more efficient than Choice A of γ in obtaining a rapidly convergent Neumann series of the auxiliary equation after introducing one subtraction in the kernel. To obtain similar convergence with Choice A two subtractions are usually needed. So we have shown numerical results for the choice B of the auxiliary function only. The convergence is quite satisfactory. The original Neumann series in both these cases were diverging very strongly the rate of divergence μ being 35 and 63, respectively. After one subtraction the series converges satisfactorily, the ratio between successive terms in the Neumann series being of the order of 0.25. Clearly if the original Neumann series diverges at a much faster rate or if the original kernel has a very complicated structure one subtraction in the kernel may not be sufficient to obtain a convergent Neumann series after a small number of iterations. But in such cases the technique of Padé approximation will also require a large number of iterations. At least in the present method we can introduce a second subtraction which will increase the rate of convergence.

Although some experimentation is needed to find the ideal γ in each case the present method has some advantages. Firstly, using an appropriate quadrature routine for the small number of integrals required by the present method produces accurate numerical results. On the other hand the problem of inverting large matrices has been avoided. Secondly, the present method is somewhat simpler than the method of Padé approximants because numerically it is more complicated to construct the Padé approximants using the iterative solution than to construct the solution in the present method using the iterative solution of the auxiliary equations. Finally, there are serious questions on the uniformity of convergence of the method of Padé approximants, since an $[N, M]$ Padé approximant has N poles, not all of which are related to the solution of the original equation. In the present method we use convergent iterative solution of auxiliary equations, eliminating the problem of spurious poles or singularities.

We applied the present method to some problems of interest in physics [5]. We solved Lippmann–Schwinger and Faddeev-type scattering equations for two- and three-particle systems and confirmed the above conclusions. In the case of two-nucleon scattering with a rather singular Reid soft core potential an accuracy of 0.005% was reached after eight iterations when we introduced two subtractions in the

kernel [5]. In the case of the three-nucleon scattering problem in the spin $3/2$ Amado model the same accuracy was reached after two iterations [5] whereas the method of Padé approximants needed about eight iterations to achieve similar precision. In the second problem we achieved an accuracy of 0.0001% after four iterations. It will be difficult to obtain similar accuracy in degenerate kernel methods or other methods using matrix inversions with so little numerical work. We conclude that the present method is an efficient alternative for solving integral equations.

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SADHAN K. ADHIKARI
*Departamento de Física,
Universidade Federal de Pernambuco,
50.000 - Recife, PE, Brazil*